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A Theorem on Band Anticongruence on Ordered Semigroup

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Abstract

Existence of a band anticongruence on order semigroup S is realized by a special family of strongly extensional completely prime subsets of S.

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1 Introduction

Our setting is Bishop's constructive mathematics (in sense of [1], [2] and [7]), mathematics developed with Constructive Logic (or Intuitionistic Logic ([13])) - logic without the Law of Excluded Middle $P \vee \neg P$. We have to note that 'the crazy axiom' $P \Longrightarrow (P \Longrightarrow Q)$ is included in the Constructive Logic. Precisely, in Constructive Logic the 'Double Negation Law' $P \Longleftrightarrow \neg\neg P$ does not hold but the following implication $P \Longrightarrow \neg\neg P$ holds even in Minimal Logic. In Constructive Logic 'Weak Law of Excluded Middle' $\neg P \vee \neg\neg P$ does not hold, too. It is interesting that in Constructive Logic the following deduction principle $A \vee B, A \vdash B$ holds, but this is impossible to prove it without 'the crazy axiom'. Any notion in Bishop's constructive mathematics has positive defined symmetrical pair since Law of Excluded Middle does not hold in Constructive Logic. Our intention is development of these symmetrical notions

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and their compatibility with so-called the 'first notions' in Semigroup Theory. As the first, semigroup S is equipped with diversity relation compatible with the equality, and, the second, the internal semigroup operation is total extensional and strongly extensional function from $S \times S$ to S. Let $(S, =, \neq)$ be a set (in the sense of books [1], [2], and [7]), where "=" is an equality and "$\neq$" is a binary relation on X which satisfies the following properties:

$$\neg(x \neq x), \;\; x \neq y \Longrightarrow y \neq x, \;\; x \neq y \wedge y = z \Longrightarrow x \neq z$$

called *diversity relation* on S. Following Heyting, if the relation "$\neq$" satisfies the following implication

$$x \neq z \Longrightarrow (\forall y \in S)(x \neq y \vee y \neq z),$$

we say that it is an *apartness*. The semigroup operation "$\cdot$" is strongly extensional in the next sense

$$(\forall a, b, x, y \in S)((ay \neq by \Longrightarrow a \neq b) \wedge (xa \neq xb \Longrightarrow a \neq b)).$$

Let Y be a subset of S and let $x \in S$. Following Bridges, by $x \bowtie Y$ we denote $(\forall y \in Y)(y \neq x)$ and by Y^C we denote subset $\{x \in S : x \bowtie Y\}$ - the strong complement of Y in S ([13]). The subset Y of S is *strongly extensional* ([13]) in S if and only if

$$y \in Y \Longrightarrow y \neq x \vee x \in Y.$$

Let α, β be relations on S. The *filed product* ([9], [11], [12]) of α and β is the relation defined by

$$\beta * \alpha = \{(x, z) \in S \times S : (\forall y \in S)((x, y) \in \alpha \vee (y, z) \in \beta)\}.$$

For $n \geq 2$, let ${}^n\alpha = \alpha * ... * \alpha$ (n factors). Put ${}^1\alpha = \alpha$. By $c(\alpha)$ we denote the intersection $c(\alpha) = \bigcap_{n \in N} {}^n\alpha$. The relation $c(\alpha)$ is a cotransitive relation on S, by Theorem 0.4 of the paper [9], called *cotransitive internal fulfillment* of the relation α.

A relation q on S is a *coequality relation* on S ([8], [10]) if and only if

$$q \subseteq \neq \text{(consistensy)}, \; q^{-1} = q \text{ (symmetric) and } q \subseteq q * q \text{ (cotransitivity)}.$$

In this case we can construct the following factor-set $S/q = \{aq : a \in S\}$ with:

$$aq =_1 bq \Longleftrightarrow (a, b) \bowtie q, \;\; aq \neq_1 bq \Longleftrightarrow (a, b) \in q.$$

A subset T f S is a *completely prime subset* of S ([3]) if and only if

$$(\forall x, y \in S)(xy \in T \Longrightarrow x \in T \vee y \in T);$$

T is a *consistent subset* of S ([3], [4]) if and only if

$$(\forall x, y \in S)(xy \in T \Longrightarrow x \in T \wedge y \in T).$$

Let q be a coequality relation on semigroup S. For q we say that it is *anticongruence* on S ([8], [11], [12]) if and only if

$$(\forall a, b, x, y \in S)((ax, by) \in q \Longrightarrow (a, b) \in q \vee (x, y) \in q).$$

If q is anticongruence on semigroup S, then the strong complement q^C of q is a congruence on the semigroup S compatible with q. We can construct semigroups S/q with

$$aq =_1 bq \Longleftrightarrow (a, b) \bowtie q, \;\; aq \neq_1 bq \Longleftrightarrow (a, b) \in q, \;\; aq \cdot bq = (ab)q.$$

Let us remind oneself of some standard notions and notations about relations and functions: For relation $\theta \subseteq S \times S$ we say ([10]) that it is an *anti-order relation* on semigroup S if and only if:

$$\theta \subseteq \neq, \;\; \neq \subseteq \theta \cup \theta^{-1} \text{ (linearity)},$$

and compatible with the semigroup operation:

$$(\forall a, b, x, y \in S)(((ay, by) \in \theta \Longrightarrow (a, b) \in \theta) \wedge ((xa, xb) \in \theta \Longrightarrow (a, b) \in \theta)).$$

Relation "$\leq$" and "θ" are *compatible* if

$$(\forall a, b \in S)\neg(a \leq b \wedge a\theta b)$$

holds. A mapping $\varphi : S \longrightarrow T$ is *strongly extensional* if $(\forall x, x' \in S)(\varphi(x) \neq_T \varphi(x') \Longrightarrow x \neq_S x')$; φ is an *embedding* if and only if $(\forall x, x' \in S)(x \neq_S x' \Longrightarrow \varphi(x) \neq_T \varphi(x'))$. If $\varphi : S \longrightarrow T$ is a strongly extensional mapping between sets with apartnesses, then the sets $Ker\varphi = \{(x, x') \in S \times S : \varphi(x) =_T \varphi(x')\}$ and $Antiker\varphi = \{(x, x') \in S \times S : \varphi(x) \neq_T \varphi(x')\}$ are compatible equality an coequality relation on S. Besides, mapping f is *antiorder isotone* if $x\theta_S y \Longrightarrow f(x)\theta_T f(y)$ holds; f is *antiorder reverse isotone* if $f(x)\theta_T f(y) \Longrightarrow x\theta_S y$ holds.

Semigroups with apartnesses were first defined and studied by A. Heyting. After that, P. T. Johnstone, J. C. Mulvey,F. Richman, R. Mines, D. A. Romano, W. Ruitenburg, A. S. Troelstra and D. van Dalen have worked on this topic.

In [3] a construction of a coequality relation q on a semigroup $(S, =, \neq, \cdot, \leq, \theta)$ with apartness order under a pair of an order "$\leq$" and an anti-order "θ" relations such that q is a band anticongruence on ordered semigroup S

is given. (Definition of band anticongruence on semigroup S ordered under a pair of these relations is given below.)

For undefined notions and notations of semigroup theory we refer to [3], [4] and of items in Constructive mathematics we refer to books [1], [2], [7] and [13] and to papers [8]-[12].

2 Preliminaries

Recall that a semigroup S is called *band* if $a^2 = a$, for all $a \in S$. As the first, we have the following statements:

Lemma 1 ([3], Lemma 1): *Relation θ on band semigroup S, defined by*

$$(a, b) \in \theta \Longleftrightarrow a \neq ab \ \vee \ a \neq ba$$

is an anti-order relation on S.

If a semigroup S is a band, then the set S endowed with the relations "$\leq$" and "θ" defined by $x \leq y \Longleftrightarrow x = xy = yx$ and $x\theta y \Longleftrightarrow x \neq xy \vee, x \neq yx$ for every x,y of S. Let we note that if $(S, =, \neq, \cdot)$ is a band, then $(S, =, \neq, \cdot, \leq, \theta)$ is not an ordered semigroup, in general (unless in the special case when the multiplication on S is commutative).

Now, following the classical definition in [6], we will define band anticongruence on semigroup $(S, =, \neq, \cdot, \leq, \theta)$ ordered under compatible order "$\leq$" and anti-order "θ": An anticongruence q on ordered semigroup $(S, =, \neq, \cdot, \leq, \theta)$ is a *band anticongruence* on S if and only if

$$x \leq y \Longrightarrow (x, xy) \bowtie q \ \wedge \ (x, yx) \bowtie q, \ \textit{and}$$

$$(a, ab) \in q \ \vee \ (a, ba) \in q \Longrightarrow (a, b) \in \theta.$$

As it is seen, our definition is different to the classical definition in paper [6] because we dealing in the Constructive mathematics and, except that, must there some connection between the anticongruence and the antiorder relation on such semigroup.

In the following some conditions about existence of band anti-congruence on ordered semigroup under pair of order and anti-order are shown.

Theorem 2 ([3], Theorem 3) *Let $(S, =, \neq, \cdot, \leq, \theta)$ be an ordered semigroup. The following are equivalent:*
(1) There exists a band anticongruence on S.
(2) There exists a band $(B, =_1, \neq_1, \circ, \leq_1, \theta_1)$ and a mapping $\pi : S \longrightarrow B$ which is strongly extensional order isotone and anti-order reverse isotone surjective homomorphism such that $\pi^{-1}(\{a\})$ is a strongly extensional subsemigroup of S and the following implication

$$t \in \pi^{-1}(\{a\}) \wedge y \in S \Longrightarrow ((ty \in \pi^{-1}(\{a\}) \wedge yt \in \pi^{-1}(\{a\})) \vee t\theta y)$$

holds for every $a \in B$.
(3) There exists a band $(B, =_1, \neq_1, \circ)$ and a family $\boldsymbol{R} = \{S_b : b \in B\}$ of strongly extensional subsemigroups of S, such that
(3.1) $S_a \cap S_b = \emptyset$ for all $a, b \in B$, $a \neq_1 b$
(3.2) $S = \bigcup_{b \in B} S_b$
(3.3) $S_a S_b \subseteq S_{a \circ b}$ for all $a, b \in B$
(3.4) If $a, b \in B$ such that $S_a \cap (S_b] \neq \emptyset$, then $a =_1 a \circ b =_1 b \circ a$;
(3.5) $t \in S_a \wedge y \in S \Longrightarrow ((ty \in S_a \wedge yt \in S_a) \vee t\theta y)$ for every a of S.

Let us point out as distinguished from the classical case ([6]) here we have an additional property (3.5) of the family **R**.

3 The main results

Let $S = (S, =, \neq, \cdot, \leq, \theta)$ is an ordered semigroup under an order and an anti-order relations. The point (3) of above theorem is asking of existence a special family of strongly extensional subsemigroups of S. As distinguished from that, in this paper existence of band anticongruence on order semigroup S is realized by a special family of strongly extensional completely prime subsets of S.

Theorem 3 *Let $(S, =, \neq, \cdot, \leq, \theta)$ be an ordered semigroup. If q is a is band anticongruence on semigroup S, then there exists a band $(\boldsymbol{B}, =_1, \neq_1, \circ)$ and a family $\boldsymbol{R} = \{T_b : b \in \boldsymbol{B}\}$ of strongly extensional completely prime subsets of S , such that*
(1) $(\forall A \in \boldsymbol{B})(\exists x \in S)(x \bowtie A)$,
(2) $(\forall y \in S)(\exists B \in \boldsymbol{B})(y \bowtie B)$,
(3) $(\forall A, B \in \boldsymbol{B})(A \neq_1 B \Longrightarrow A \cup B = S)$,
(4) $(\forall A \in \boldsymbol{B})(t \bowtie A \wedge y \in S \Longrightarrow ((ty \bowtie A \wedge yt \bowtie A) \vee, t\theta y)$.
Opposite, if the conditions (1)-(4) hold and, besides, the following condition
(5) $(\forall C)(\forall u, v \in S)(uv \bowtie C \Longrightarrow u \bowtie C \wedge v \bowtie C)$,
holds also, then there exists a band anticongruence on S.

Proof:

$\Longrightarrow$ Let q be a band anticongruence on S. For $x \in S$ let we put $xq = \{y \in S : (x,y) \in q\}$, the class generated by x, and $S/q = \{xq : x \in S\}$ factor-semigroup. It is well-known that for a coequality relation q on set S there the family of strongly extensional subsets of S which satisfies conditions (1)-(3) (See, for example, Theorem 0.1 in [12]).
(a) $(\forall x \in S)((x,x^2) \bowtie q \wedge (x^2,x) \bowtie q)$ and $(\forall x \in S)(xq \cdot xq =_1 xq)$. In fact, if $a \in S$, then $x \leq x$, and $(x,x^2) \bowtie q$. Since q is symmetric, we also have $(x^2,x) \bowtie q$. Therefore, immediately $(\forall x \in S)(x^2q =_1 xq)$ holds.
(b) S/q is a band. Indeed, as we have already seen $(S/q, =_1, \neq_1, \cdot)$ is a semigroup, and by (a), it is a band.
(c) The class xq, generated by the element x of S, is a strongly extensional completely prime right subset of S for all x in S. Indeed, let $x \in S$. Clearly that $xq \subset S$ because $x \bowtie xq$. Let $uv \in xq$. Then, $(x,uv) \in q$. Thus, $(x,x^2) \in q$ or $(x^2,uv) \in q$. Since case $(x,x^2) \in q$ is impossible, we have $(x,u) \in q$ or $(x,v) \in q$ because q is an anticongruence on S.
(d) If $x \leq y$, then $(xy,yx) \bowtie q$. Indeed, let $x \leq y$. Since q is a band anticongruence on S, we have $(x,xy) \bowtie q$ and $(x,yx) \bowtie q$. Let (u,v) be arbitrary element of q. Then $(u,xy) \in q$ or $(xy,x) \in q$ or $(x,yx) \in q$ or $(yx,v) \in q$. Hence $u \neq ab$ or $ba \neq v$ because cases $(xy,x) \in q$ and $(x,yx) \in q$ are impossible. So, $(u,v) \neq (xy,yx)$.
(f) If q is a band anticongruence on ordered semigroup $(S,=,\neq,\cdot,\leq,\theta)$ then, as we have already seen $(S/q,=_1,\neq_1,\cdot)$ is a band. So, the set S/q, with the relations "$\leq_1$" and "θ_1" on S/q defined by:

$$xq \leq_1 yq \Longleftrightarrow (xq =_1 xq \cdot yq \wedge xq =_1 yq \cdot xq),$$

$$xq\theta_1 yq \Longleftrightarrow (xq \neq_1 xq \cdot yq \vee xq \neq_1 yq \cdot xq),$$

is ordered set under compatible order and antiorder relation. Moreover, since q is a band anticongruence, we have

$$x \leq y \Longrightarrow (x,xy) \bowtie q \wedge (x,yx) \bowtie q$$

$$\Longleftrightarrow xq =_1 xyq \wedge xq =_1 yxq$$

$$\Longleftrightarrow xq \leq_1 yq;$$

$$xq\theta_1 yq \Longleftrightarrow xq \neq_1 xq \cdot yq \vee xq \neq_1 yq \cdot xq$$

$$\Longleftrightarrow (x,xy) \in q \vee (x,yx) \in q$$

$$\Longrightarrow x\theta y.$$

Therefore, the natural epimorphism $\pi : S \longrightarrow S/q$ is order isotone and antiorder reverse isotone homomorphism.

(g) $t \bowtie [x] \wedge y \in S \Longrightarrow ((ty \bowtie [x] \wedge yt \bowtie [x]) \vee t\theta y)$. In fact, let u be an arbitrary element of $[x]$. Then $(u,x) \in q$. Thus, $(u,ty) \in q$ or $(ty,t) \in q$ or $(t,x) \in q$. Hence $u \neq ty$ or $t\theta y$, because $(t,x) \in q$ is impossible. So, the implication $t \bowtie [x] \wedge y \in S \Longrightarrow (ty \bowtie [x] \vee t\theta y)$ holds. Analogously we show the implication $t \bowtie [x] \wedge y \in S \Longrightarrow (yt \bowtie [x]) \vee t\theta y)$.

$\Longleftarrow$ Let us define relation q on S by $(x,y) \in q \Longleftrightarrow (\exists A \in \mathbf{B})(x \bowtie A \wedge y \in A)$.

(I) First, q is a coequality relation on S. Indeed:
(i) Relation q is consistent: It is clearly that $q \subseteq \neq$.
(ii) Relation q is symmetric: Let $(x,y) \in q$, i.e. let there exists an element A of $\mathbf{B}$ such that $x \bowtie A \wedge y \in A$. By (2) there exists B of $\mathbf{B}$ such that $y \bowtie B$. From $A \neq_1 B$ we conclude $S = A \cup B$. So, $x \in B$ holds. Thus, $(y,x) \in q$.
(iii) Relation q is cotransitive: Suppose that $(x,z) \in q$ and let y be an arbitrary element of S. Then there exist elements A and C of $\mathbf{B}$ such that $x \bowtie A \wedge z \in A$ and $z \bowtie C \wedge x \in C$. By (2) there exists an element B of $\mathbf{B}$ such that $y \bowtie B$. Except that, from $y \in S = A \cup C$ we have $y \in A$ or $y \in C$. Now, we have possibilities:
$x \bowtie A \wedge z \in A$ and $y \in A \Longrightarrow (x,y) \in q$;
$x \bowtie A \wedge z \in A$ and $y \in C \Longrightarrow z \bowtie C \wedge x \in C$ and $y \in C$
$\Longrightarrow (z,y) \in q$.
So, we conclude that q is a coequality relation on S.

(II) Third, let $(a,ab) \in q$. Then there exist elements α of $\mathbf{B}$ such that $(a \bowtie \alpha \wedge ab \in \alpha)$ (and there exists element β of $\mathbf{B}$ such that $(ab \bowtie \beta \wedge a \in \beta)$). By (4) we have the implication: $a \bowtie \alpha \wedge b \in S \Longrightarrow (ab \bowtie \alpha \vee a\theta b)$. Since the cases $ab \in \alpha$ and $ab \bowtie \alpha$ are impossible, we conclude $a\theta b$. So, the implication $(a,ab) \in q \Longrightarrow a\theta b$ holds. The implication $(a,ab) \in q \Longrightarrow a\theta b$ is proved analogously.

(III) Let x,y be elements of S such that $x \leq y$, and let (u,v) be an arbitrary element of q. Then, $(u,x) \in q \vee (x,xy) \in q \vee (xy,v) \in q$. Suppose that $(x,xy) \in q$. Then there exists element α (and β) of $\mathbf{B}$ such that $(x \bowtie \alpha \wedge xy \in \alpha)$ (and $(xy \bowtie \beta \wedge x \in \beta)$) holds.
From $xy \in \alpha$ we conclude that $x \in \alpha$ or $y \in \alpha$ because α is a strongly extensional and completely prime subset of S. Thus, $y \in \alpha$ since $x \in \alpha$ is impossible. By (5), we have $xy \bowtie \alpha \vee x\theta y$. Both cases drive us in contradiction. So, must be $(u,v) \neq (x,xy)$. Therefore, the implication $x \leq y \Longrightarrow (x,xy) \bowtie q$ is valid. Validity of another implication $x \leq y \Longrightarrow (x,yx) \bowtie q$ we prove analogously.

(IV) Finally, let x, y and u be arbitrary elements of semigroup S such that $(ux, yu) \in q$. Then there exists a subset C of S such that $ux \bowtie C \wedge uy \in C$. From $uy \in C$ we have $u \in C$ or $y \in C$. The second, by (5), holds $ux \bowtie C \Longrightarrow (u \bowtie C \wedge x \bowtie C)$. Thus
$(ux, uy) \in q \Longleftrightarrow (\exists C \in \mathbf{B})(ux \bowtie C \wedge uy \in C)$
$\Longrightarrow (\exists C \in \mathbf{B})((u \bowtie C \wedge x \bowtie C) \wedge (u \in C \vee y \in C))$
$\Longrightarrow (\exists C \in \mathbf{B})((u \bowtie C \wedge x \bowtie C \wedge u \in C) \vee (u \bowtie C \wedge x \bowtie C \wedge y \in C))$
$\Longrightarrow (\exists C \in \mathbf{B})(u \bowtie C \wedge x \bowtie C \wedge y \in C)$
$\Longrightarrow (x, y) \in q$.

For the implication $(xu, yu) \in q \Longrightarrow (x, y) \in q$ we have analogous proof. □

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